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THE THEORY OF PITCH INVARIANTS AND THE THEORY OF CHIASTIC HOMOGRAPHY, BEING THE TENTH MEMOIR ON THE "THEORY OF SCREWS."

By SIR ROBERT S. BALL, LL.D., F.R.S., Lowndean Professor of Astronomy and Geometry in Cambridge.

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XIV.

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[Read January 22, 1894.]

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I.—Introduction.

In various parts of the Theory of Screws I have had occasion to consider the nature of those functions of the screw co-ordinates which possess the property of pitch invariants. Let $\theta_1, \ldots \theta_6$ be the six co-ordinates of a screw, and let $f(\theta_1, \ldots \theta_6)$ be an homogeneous function of these quantities. It will generally happen that the value of the function depends on the pitch of θ. Obviously, however, there must be cases in which the pitch of θ is immaterial to the value of the function. For example, an expression in terms of the co-ordinates of two screws which signified the cosine of the angle between them must be the same whatever the pitch of either screw may be. In this case the function is what we may term a pitch invariant. There are several cases in the Theory of Screws where it is material to examine the character of functions of the co-ordinates in their relations to pitch. I have,

therefore, thought it worth while to undertake a general investigation of the question. The result has been to obtain certain theorems which appear to be of sufficient interest to justify my making a communication on the subject to the Royal Irish Academy.

Let us suppose p_θ to be the pitch of θ, then we have first to ascertain the co-ordinates $\phi_1, \ldots \phi_6$ of the screw ϕ which lies on the same axis as θ, but has a pitch

$$p_\theta + x.$$

We assume, as usual, that the screws of reference are co-reciprocal, and that their pitches are $p_1, \ldots p_6$. Let $\epsilon_1, \ldots \epsilon_6$ be the angles which θ makes with the six screws of reference, and let $d_1, \ldots d_6$ be the several shortest distances between θ and the screws of reference. Then, by the known principles of the Theory of Screws, we have

$$\theta_1 = \frac{(p_\theta + p_1)\cos\epsilon_1 - d_1 \sin\epsilon_6}{2p_1},$$

&c., &c.

$$\theta_6 = \frac{(p_\theta + p_6)\cos\epsilon_6 - d_6 \sin\epsilon_6}{2p_6},$$

also, since θ and ϕ are on the same axes,

$$\phi_1 = \frac{(p_\theta + x + p_1)\cos\epsilon_1 - d_1 \sin\epsilon_1}{2p_1},$$

&c., &c.

$$\phi_6 = \frac{(p_\theta + x + p_6)\cos\epsilon_6 - d_6 \sin\epsilon_6}{2p_6}.$$

Whence we obtain the expressions obtained in another manner in the "Theory of Screws," p. 100.

$$\phi_1 = \theta_1 + \frac{x}{2p_1}\cos\epsilon_1,$$

&c., &c.

$$\phi_6 = \theta_6 + \frac{x}{2p_6}\cos\epsilon_6.$$

If a function f be a pitch invariant, then it is obvious that the substitution of $\phi_1, \ldots \phi_6$ for $\theta_1, \ldots \theta_6$ must leave it unaltered in value; in other words, the several coefficients of x in the following expression must vanish identically:—

$$f + x\Delta f + \frac{x^2}{1 \cdot 2} \Delta^2 f +, \text{ \&c.},$$

when Δ is the operation,

$$\frac{\cos \epsilon_1}{2p_1} \cdot \frac{d}{d\theta_1} + \ldots + \frac{\cos \epsilon_6}{2p_6} \cdot \frac{d}{d\theta_6},$$

this must be true for every set of angles $\epsilon_1, \ldots \epsilon_6$, that can be made by a ray with the six screws of reference.

II.—Linear Pitch Invariant Functions.

We propose to investigate the linear functions of the six co-ordinates of a screw which possess the property that they remain unaltered notwithstanding an alteration in the pitch of the screw which the co-ordinates denote. It will first be convenient to demonstrate a general theorem which introduces a property of the six screws of a co-reciprocal system.

Let α be any screw. As usual we denote the co-ordinates of this screw by $\alpha_1, \ldots \alpha_6$. We employ the symbols $a_1, \ldots a_6$ to express the angles which the axis of the screw makes with the six co-reciprocal screws of reference. We shall similarly use β_1, &c., β_6 for the co-ordinates of the screw β, and $b_1, \ldots b_6$ will be the angles which its axis makes with the screws of reference.

The virtual coefficient of two screws is represented by half the expression

$$(p_\alpha + p_\beta) \cos \theta - d \sin \theta,$$

where p_α and p_β are the pitches, θ is the angle between the two screws, and d the shortest perpendicular distance between them. The pitches only enter into this expression as regards their sum; and, consequently, if p_α be changed into $p_\alpha + x$, and p_β be changed into $p_\beta - x$, the virtual coefficient must remain unaltered whatever x may be.

We know, however, that the virtual coefficient admits of representation in the form

$$p_1\alpha_1\beta_1 + \&c. + p_6\alpha_6\beta_6.$$

To augment the pitch of α by x, we substitute for α_1, α_2, &c., the several values

$$\alpha_1 + \frac{x}{2p_1}\cos a, \quad \alpha_2 + \frac{x}{2p_2}\cos a_2, \ \&c.$$

Similarly, to diminish the pitch of β by x, we substitute for β_1, β_2, &c., the several values of

$$\beta_1 - \frac{x}{2p_1}\cos b_1, \quad \beta_2 - \frac{x}{2p_2}\cos b_2, \ \&c.$$

With this change the virtual coefficient, as above expressed, becomes

$$\Sigma p_1\left(\alpha_1 + \frac{x}{2p_1}\cos a_1\right)\left(\beta_1 - \frac{x}{2p_1}\cos b_1\right),$$

or,

$$\Sigma p_1\alpha_1\beta_1 + \frac{x}{2}(\beta_1\cos a_1 + \beta_2\cos a_2 + \&c. - \alpha_1\cos b_1 - \alpha_2\cos b_2 - \&c.)$$

$$-\frac{x^2}{4}\left(\frac{\cos a_1\cos b_1}{p_1} + \frac{\cos a_2\cos b_2}{p_2} + \&c. + \frac{\cos a_6\cos b_6}{p_6}\right).$$

But we have already shown that such a change must, for all values of x, be devoid of effect upon the virtual coefficient. It therefore follows that the coefficients both of x and of x^2 in the expressions just written must be zero. Hence we obtain the two following properties :—

$$0 = (\beta_1\cos a_1 + \&c. + \beta_6\cos a_6) - (\alpha_1\cos b_1 + \&c. + \alpha_6\cos b_6),$$

$$0 = \frac{\cos a_1\cos b_1}{p_1} + \&c. \ \&c. + \frac{\cos a_6\cos b_6}{p_6}.$$

The latter of these two formulæ is the important one for our present purpose. It will be noted that though the two screws, α and β, are completely arbitrary, yet the six direction cosines of α with regard to the screws of reference, and the six direction cosines of β with regard to the same screws of reference, must be connected by this curious relation. Of course this equation is only true when the six screws of reference are *co-reciprocal.*

In the more general case the equivalent identity would be of a much more complicated type.

This formula provides a demonstration of the following theorem :—

Let $h_1, \ldots h_6$ be the direction angles of any ray whatever with regard to six co-reciprocal screws of reference, then the function

$$U = \alpha_1 \cos h_1 + \ldots + \alpha_6 \cos h_6$$

is a pitch invariant.

For, if we augment the pitch of α by x, we have to write for $\alpha_1 \ldots \alpha_6$ the expressions

$$\alpha_1 + \frac{x}{2p_1} \cos a_1, \ldots \alpha_6 + \frac{x}{2p_6} \cos a_6,$$

and then U becomes

$$\alpha_1 \cos h_1 + \ldots + \alpha_6 \cos h_6$$

$$+ \frac{x}{2}\left(\frac{\cos a_1 \cos h_1}{p_1} + \ldots + \frac{\cos a_6 \cos h_6}{p_6}\right);$$

but from what we have just proved, the co-efficient of x is zero, and hence we see that

$$\alpha_1 \cos h_1 + \ldots + \alpha_6 \cos h_6,$$

remains unchanged by any alteration in the pitch of α.

If we take three mutually rectangular screws, α, β, γ, then we have the three pitch invariants—

$$L = \theta_1 \cos a_1 + \ldots + \theta_6 \cos a_6.$$
$$M = \theta_1 \cos b_1 + \ldots + \theta_6 \cos b_6.$$
$$N = \theta_1 \cos c_1 + \ldots + \theta_6 \cos c_6.$$

It is obvious that any linear function of L, M, N, such as

$$fL + gM + hN,$$

is a pitch invariant.

We can further show that this is the most general type of linear pitch invariant.

For the conditions under which the general linear function $A_1\,\theta_1 + \ldots + A_n\,\theta_n$ shall be a pitch invariant are that equations of the type

$$\frac{A_1 \cos a_1}{p_1} + \ldots + \frac{A_6 \cos a_6}{p_6} = 0\,;\ \&c.$$

shall be satisfied for all possible rays.

Though these equations are infinite in number, yet they are only equivalent to three independent equations; in other words, if these equations are satisfied for three rays, a, b, c, which, for convenience, we may take to be rectangular, then they are satisfied for every ray.

For, take a ray ϵ, which makes direction-angles λ, μ, ν with a, b, c, then we have

$$\cos \epsilon_1 = \cos \lambda \cos a_1 + \cos \mu \cos b_1 + \cos \nu \cos c_1,$$

$$\&c.\ \&c.$$

$$\cos \epsilon_6 = \cos \lambda \cos a_6 + \cos \mu \cos b_6 + \cos \nu \cos c_6.$$

Hence

$$\frac{A_1 \cos \epsilon_1}{p_1} + \ldots + \frac{A_6 \cos \epsilon_6}{p_6} = \cos \lambda\, \Sigma\, \frac{A_1 \cos a_1}{p_1} + \cos \mu\, \Sigma\, \frac{A_1 \cos b_1}{p_1} + \cos \nu\, \Sigma\, \frac{A_1 \cos c_1}{p_1}.$$

If, therefore, we have

$$\Sigma\, \frac{A_1 \cos a_1}{p_1} = 0\,;\quad \Sigma\, \frac{A_1 \cos b_1}{p_1} = 0\,;\quad \Sigma\, \frac{A_1 \cos c_1}{p_1} = 0.$$

then, for every ray, we shall have

$$\Sigma\, \frac{A_1 \cos \epsilon_1}{p_1} = 0.$$

It thus follows that the coefficients of a linear function which possesses the property of a pitch invariant must be subjected to three conditions. There are accordingly only three coefficients left disposable in the most general type of linear pitch invariant. Now,

$$fL + gM + hN$$

is a pitch invariant which contains three disposable quantities, f, g, h; it

therefore represents the most general form of linear function which possesses the required property.

We have thus solved the problem of finding a perfectly general expression for the linear pitch invariant function of the co-ordinates of a screw.

It is convenient to take the three fundamental rays as mutually rectangular; but it is, of course, easy to show that any linear pitch invariant can be expressed in terms of three pitch invariants unless their determining rays are coplanar. We may express the result thus:—Let L, M, N, O be four linear pitch invariants, no three of which have coplanar determining rays. Then it is always possible to find four parameters, λ, μ, ν, ρ, such that the following equation shall be satisfied identically:—

$$0 = \lambda L + \mu M + \nu N + \rho O.$$

The nature of the pitch invariant function can be otherwise seen. It is well known that in the composition of two or more twist velocities we discover the direction of the resultant screw and the magnitude of the resultant twist velocity by proceeding as if the twist velocities were vectors. Neither the pitches of the component screws nor their situations affect the magnitude of the resultant twist velocity or the direction of the resultant screw. This principle is, of course, an immediate consequence of the law of composition of twist velocities by the cylindroid.

Let any ray α make an angle λ with the ray θ, and angles a_1, &c., a_6 with the six screws of reference. The twist velocity $\dot{\theta}$ on θ if resolved on α has a component $\dot{\theta} \cos \lambda$. This must be equal to the sum of the several components $\dot{\theta}_1$, $\dot{\theta}_2$, &c., resolved on α; whence we have

$$\dot{\theta} \cos \lambda = \dot{\theta}_1 \cos a_1 + \ldots + \dot{\theta}_6 \cos a_6.$$

If we make $\dot{\theta} =$ unity, we obtain

$$\cos \lambda = \theta_1 \cos a_1 + \ldots + \theta_6 \cos a_6.$$

This at once gives a geometrical meaning to the pitch invariant. It is simply the cosine of the angle between the screws θ and α. As, of course, the pitch is not involved in the notion of this angle, it is, indeed, obvious that the expression for any function of the angle must be a pitch invariant.

We now see the meaning of the equation obtained by equating the pitch invariant to zero. If we make

$$0 = \theta_1 \cos a_1 + \ldots + \theta_6 \cos a_6$$

it follows that α and θ must be at right angles. The equation therefore signifies the locus of all the screws that are at right angles to α.

The two equations—

$$\theta_1 \cos a_1 + \ldots + \theta_6 \cos a_6 = 0$$
$$\theta_1 \cos b_1 + \ldots + \theta_6 \cos b_6 = 0,$$

denote the screws perpendicular to the two directions of α and β. In other words, these two equations define all the screws perpendicular to a given plane.

Let us now take the case where α, β, γ are three rectangular screws, and examine the conditions satisfied by $\theta_1, \ldots \theta_6$ when subjected to the three following equations :—

$$\theta_1 \cos a_1 + \ldots + \theta_6 \cos a_6 = 0$$
$$\theta_1 \cos b_1 + \ldots + \theta_6 \cos b_6 = 0$$
$$\theta_1 \cos c_1 + \ldots + \theta_6 \cos c_6 = 0.$$

The screws which satisfy these conditions must all be perpendicular to the three directions of α, β, γ. At first this seems impossible; for how could any ray be perpendicular to each of three rays which were themselves mutually rectangular? This is only possible if the rays denoted by $\theta_1, \ldots \theta_6$ are lines at infinity.

It follows that the three equations, $L = 0$; $M = 0$; $N = 0$, obtained by equating the three fundamental pitch invariants to zero, must express the collection of screws that are situated in the plane at infinity.

We can write the three equations in an equivalent form by the six equations

$$\theta_1 = f\frac{\cos a_1}{p_1} + g\frac{\cos b_1}{p_1} + h\frac{\cos c_1}{p_1},$$

&c., &c., &c.,

$$\theta_6 = f\frac{\cos a_6}{p_6} + g\frac{\cos b_6}{p_6} + h\frac{\cos c_6}{p_6},$$

where f, g, h are any quantities whatever; for it is obvious that, by substituting these values for θ_1, &c., θ_6 in either L, or M, or N, these quantities are made to vanish by the formulæ of the type

$$\frac{\cos a_1 \cos b_1}{p_1} + \&c. + \frac{\cos a_6 \cos b_6}{p_6} = 0.$$

We have, consequently, in the expressions just written for θ_1, &c., θ_6, the values of the co-ordinates of a screw which lies entirely in the plane at infinity.

We may here note a somewhat remarkable expression for the pitch of a screw. It is known that if α, β, γ be the direction angles of a ray, and if P, Q, R be its shortest perpendicular distances from three rectangular axes, then

$$O = P \sin \alpha \cos \alpha + Q \sin \beta \cos \beta + R \sin \gamma \cos \gamma.$$

Let η, ξ, ζ be three screws of zero pitch, which intersect at right angles, and let θ be another screw, then, if $\varpi_{\eta\theta}$ be the virtual coefficients of η and θ,

$$2\varpi_{\eta\theta} = p_\theta \cos \alpha - P \sin \alpha,$$

whence, by the theorem just mentioned, we have

$$p_\theta = 2\varpi_{\eta\theta} \cos \alpha + 2\varpi_{\xi\theta} \cos \beta + 2\varpi_{\zeta\theta} \cos \gamma.$$

Let $a_1, \ldots a_6$ be the angles made by η with the six co-reciprocal screws of reference, then

$$\cos \alpha = L = \theta_1 \cos a_6 + \ldots + \theta_6 \cos a_6,$$

and, similarly, for the two other angles,

$$\cos \beta = M = \theta_1 \cos b_1 + \ldots + \theta_6 \cos b_6,$$

$$\cos \gamma = N = \theta_1 \cos c_1 + \ldots + \theta_6 \cos c_6,$$

and $L^2 + M^2 + N^2 = 1$, whence we have for the pitch the homogeneous expression,

$$p_\theta = \frac{2L\varpi_{\eta\theta} + 2M\varpi_{\xi\theta} + 2N\varpi_{\zeta\theta}}{L^2 + M^2 + N^2}.$$

It appears from this that the three equations,

$$\varpi_{\eta\theta} = 0; \quad \varpi_{\xi\theta} = 0; \quad \varpi_{\zeta\theta} = 0,$$

indicate that θ must be one of a pencil of rays of zero pitch radiating from a point.

The equations $L = 0$; $M = 0$; $N = 0$, define a screw of indeterminate pitch.

Why all the screws in the plane at infinity should present themselves with indeterminate pitch is a question which requires some answer. The twist about such a screw, as around any other, consists, of course, of a rotation and a translation. If, however, the finite parts of the body are only to be moved through a finite distance, the amplitude of the twist must be infinitely small, for a finite rotation around an axis at infinity would, of course, imply an infinitely great displacement of parts of the body which were at finite distances. The amplitude of the rotation is therefore infinitely small, so that, if the pitch is finite, the displacement parallel to the axis of the screw is infinitely small also. It thus appears that the effect of a small twist about a screw of any finite pitch at infinity is to give the finite parts of the body two displacements, one of which is infinitely insignificant as regards the other. We can therefore overlook the displacement due to the pitch, and consequently the pitch of the screw is immaterial; in other words, in so far as the screw is the subject of our investigation, its pitch is indeterminate.

In like manner we can prove that a screw in the plane at infinity, when regarded as the seat of a wrench, must, when finite forces are considered, be regarded as possessing an indeterminate pitch. For, let the force appertaining to the wrench be of finite magnitude, then its effect on bodies at finite distances would involve a couple of infinite moment. It therefore follows that the force on the screw at infinity must be infinitely small if the effects of the wrench are finite. The moment of the couple on the screw is therefore infinitely small, nor is its magnitude increased by importation from infinity; therefore, at finite distances, the effect of the couple part of the wrench may be neglected in comparison with that of the force part of the wrench. But the pitch of the screw is only involved so far as the couple is concerned; and hence whatever be the pitch of the screw lying in the plane at infinity, its effect is inoperative so far as finite operations are concerned.

There is here a phenomenon of duality which, though full of significance in non-Euclidian space, merely retains a shred of its importance in the space of which we now treat. A displacement, such as we have been considering, may arise either from a twist about a screw of infinite pitch at an indefinite distance, or a twist about a screw of indefinite pitch at an infinite distance. I do not now further enter into this subject, for, in a future treatment of the theory in non-Euclidian space, it is expected to be fully developed.

We can now see the reason why two screws in the plane at infinity are reciprocal. It is because they are coplanar, and the pitches being indeterminate, we may so arrange that their sum is zero, *i. e.* that the screws are reciprocal.

III.—On the Relations between Impulsive Screws and Instantaneous Screws.

Although much has been written in preceding Memoirs of this series on the subject of the relations between the several pairs of impulsive screws and the corresponding instantaneous screws, yet it is only lately that I have discovered a significant point in the theory that I now set forth.

We shall first suppose the case of a perfectly free rigid body. Let α and β be two instantaneous screws, the corresponding impulsive screws being η and ξ. It is not generally possible to choose the four screws α, β and η, ξ arbitrarily. I mean that, if four screws be taken at random, it is not generally possible to adapt a rigid body of such a constitution, and in such a position, that η and ξ shall be the impulsive screws corresponding, respectively, to α and β as instantaneous screws. This will be possible, but only possible, if the four screws fulfil a condition expressed by one equation, the nature of which we shall now proceed to explain.

It might, indeed, have been foreseen that there must be some condition of the kind from the following way of viewing the subject:—

α and β being chosen arbitrarily, it will require five constants to specify η, and five more to specify ξ. It therefore appears that to apply a rigid body which shall give the required relations between the screws, there

should be ten constants disposable. But the rigid body, so far as this question is concerned, has only nine disposable co-ordinates. Three of these are the co-ordinates of its centre of inertia; three more are the directions of its principal axes through the centre of inertia; and the remaining three are the radii of gyration. The mass of the body is not significant for the question, because the velocity generated by an impulse of specified intensity is not now material. As it is therefore generally impossible to adapt a body with only nine disposable constants to a case where ten conditions have to be fulfilled, it follows that if the four screws, α, β, η, ξ, be completely independent, they cannot be two pairs of impulsive and instantaneous screws.

The nature of the condition they must satisfy will now appear.

Let us take the principal screws of inertia for the screws of reference; hen if η be the impulsive screw corresponding to α, we have for the co-ordinates of η,

$$\frac{a\alpha_1}{H}, -\frac{a\alpha_2}{H}, \frac{b\alpha_3}{H}, -\frac{b\alpha_4}{H}, \frac{c\alpha_5}{H}, -\frac{c\alpha_6}{H},$$

where $$H^2 = a^2(\alpha_1 - \alpha_2)^2 + b^2(\alpha_3 - \alpha_4)^2 + c^2(\alpha_5 - \alpha_6)^2,$$

and we also write,

$$R^2 = (\alpha_1 + \alpha_2)^2 + (\alpha_3 + \alpha_4)^2 + (\alpha_5 + \alpha_6)^2.$$

The cosine of the angle between any two screws θ and ϕ is

$$(\theta_1 + \theta_2)(\phi_1 + \phi_2) + (\theta_3 + \theta_4)(\phi_3 + \phi_4) + (\theta_5 + \theta_6)(\phi_5 + \phi_6).$$

We therefore have, if we denote by $\cos(\alpha\eta)$ the cosine of the angle between α and η,

$$\cos(\alpha\eta) = \frac{1}{HR}\left[a\alpha_1^2 - a\alpha_2^2 + b\alpha_3^2 - b\alpha_4^2 + c\alpha_5^2 - c\alpha_6^2\right]$$
$$= \frac{1}{HR}p_\alpha \,.\, R^2 = \frac{R}{H}p_\alpha$$

where p_α is the pitch of α.

Similarly, $$\cos(\beta\xi) = \frac{S}{K}p_\beta$$

where

$$S^2 = (\beta_1 + \beta_2)^2 + (\beta_3 + \beta_4)^2 + (\beta_5 + \beta_6)^2,$$

$$K^2 = a^2(\beta_1 - \beta_2)^2 + b^2(\beta_3 - \beta_4)^2 + c^2(\beta_5 - \beta_6)^2.$$

If $\varpi_{\alpha\xi}$ denote the vertical coefficients of the screws α and ξ, we have

$$\varpi_{\alpha\xi} = \frac{1}{KR}\left[a^2\alpha_1\beta_1 + a^2\alpha_2\beta_2 + b^2\alpha_3\beta_3 + b^2\alpha_4\beta_4 + c^2\alpha_5\beta_5 + c^2\alpha_6\beta_6\right],$$

$$\varpi_{\beta\eta} = \frac{1}{HS}\left[a^2\alpha_1\beta_1 + a^2\alpha_2\beta_2 + b^2\alpha_3\beta_3 + b^2\alpha_4\beta_4 + c^2\alpha_5\beta_5 + c^2\alpha_6\beta_6\right],$$

whence we obtain by elimination of K, R, H, S the remarkable result,

$$\frac{\cos(\alpha\eta)}{p_\alpha}\varpi = \frac{\cos(\beta\xi)}{p_\beta}\varpi_{\beta\eta}.$$

Notwithstanding the way this has been demonstrated it will be noticed that it is wholly independent of whatever screws of reference may have been chosen. It depends solely on the various attributes and relations of the four screws themselves. In general, if the co-ordinates of the screws can be made to satisfy this condition, it will be possible to adjust the constitution of a rigid body so that α and η, β and ξ shall stand to each other in the required relation.

It is, of course, a fundamental point in the theory of screws, that when two pairs α, η and β, ξ of impulsive and instantaneous screws are so related that α is reciprocal to ξ, then must β be reciprocal to η. The two screws, α and β, are then designated "conjugate screws of inertia."

The general relation just proved between the impulsive screw and the instantaneous screw refers to the case where the body is perfectly free. We have now to show that this relation is itself a particular case of a more general theorem which holds when the body is only partially free. The result is one of no little interest in the theory.

Let η and ξ be, as before, a pair of impulsive screws, and let α and β be the corresponding pair of instantaneous screws. Let ρ be the screw on which a reaction is contributed by the constraints at the moment when the impulsive wrench is applied on η. Let β be similarly related to ξ.

The movement of the body twisting about α is therefore the same as

if it had been free, and an impulsive wrench had been imparted about η and another about ρ, so that the following conditions are satisfied :—

$$Hp_1\ \alpha_1 = \eta''\eta_1 + \rho''\rho_1,$$

$$\&c.\ \&c.\ \&c.$$

$$Hp_n\ \alpha_6 = \eta''\eta_6 + \rho''\rho_6.$$

Multiply by $p_1\ \alpha_1, \ldots p_6\ \alpha_6$, respectively, and adding, we have

$$Hu_{\alpha\alpha} = \eta''\varpi_{\eta\alpha},$$

where

$$u_{\alpha\alpha} = p_1^2\alpha_1^2 + \ldots p_6^2\alpha_6^2;$$

because, as ρ belongs to the reciprocal system, we must have

$$\varpi_{\rho\alpha} = 0.$$

Similarly, if we multiplied the six equations by $p_1\beta_1, \ldots p_6\beta_6$, respectively, and added, we should get, since ρ is reciprocal to β,

$$Hu_{\alpha\beta} = \eta''_{\eta\beta},$$

where

$$u_{\alpha\beta} = p_1^2\alpha_1\beta_1 + \ldots + p_6^2\alpha_6\beta_6.$$

Eliminating H and η'' we have the extremely concise result,

$$u_{\alpha\alpha}\,\varpi_{\beta\eta} = u_{\alpha\beta}\,\varpi_{\alpha\eta}.$$

In a similar way we can deal with the pair of screws, β and ξ, and by eliminating σ, the reaction of the constraints in this case, we obtain the result

$$u_{\beta\beta}\,\varpi_{\alpha\xi} = u_{\beta\alpha}\,\varpi_{\beta\xi}.$$

Finally, from these two equations we can eliminate $u_{\alpha\beta}$, and we obtain the important result

$$\frac{\varpi_{\alpha\eta}\,\varpi_{\alpha\xi}}{u_{\alpha\alpha}} = \frac{\varpi_{\beta\eta}\,\varpi_{\beta\xi}}{u_{\beta\beta}}.$$

This is a result of a perfectly general character, connecting any two pairs of impulsive and instantaneous screws α, η and β, ξ. It holds whether the body be free or constrained in any way whatever. If the body be perfectly free, then it is easy to show that it reduces to the form we have already found.

We may give another proof of this formula as follows:—

From the theory of impulsive and instantaneous screws in an n-system we know that, if $\alpha_1, \ldots \alpha_n$ be the co-ordinates of an instantaneous screw, then the co-ordinates $\eta_1 \ldots \eta_n$ of the reduced impulsive screw may be determined by equations, as follows:—

$$H\eta_1 = \frac{u_1^2}{p_1}\alpha_1,$$

&c. &c.

$$H\eta_n = \frac{u_n^2}{p_n}\alpha_n.$$

Multiplying severally by $p_1\,\alpha_1, \ldots p_n\,\alpha_n$, and adding, we have

$$H\varpi_{\eta\alpha} = u_{\alpha\alpha}.$$

Multiply similarly by $p_1\,\beta_1, \ldots p_n\,\beta_n$, and adding,

$$H\varpi_{\eta\beta} = u_{\alpha\beta}.$$

Eliminating H, we find

$$u_{\alpha\beta}\,\varpi_{\eta\alpha} = u_{\alpha\alpha}\,\varpi_{\eta\beta}.$$

A physical demonstration of this formula can be easily given. Let α, β be the two screws which correspond, as instantaneous screws, to η and ξ, as impulsive screws.

Let us take on the cylindroid α, β, a screw θ, which is conjugate to α with respect to inertia. Then, by the known principles of the theory of screws, the screw θ so defined must be reciprocal to η.

Hence

$$p_1\,\eta_1\,\theta_1 + \ldots + p_n\,\eta_n\,\theta_n = 0.$$

As, however, α and θ are conjugate, we have

$$u_1^2\,\alpha_1\,\theta_1 \ldots + u_n^2\,\alpha_n\,\theta_n;$$

also, since θ is co-cylindroidal with α and β, we have

$$\theta_1 = \lambda\alpha_1 + \mu\beta_1;\ \ldots\ \theta_n = \lambda\alpha_n + \mu\beta_n.$$

Substituting these in the two previous equations, we get

$$\lambda\varpi_{\eta\alpha} + \mu\varpi_{\eta\beta} = 0;$$

$$\lambda u_{\alpha\alpha} + \mu u_{\alpha\beta} = 0;$$

whence, as before,

$$u_{\alpha\alpha}\,\varpi_{\beta\eta} - u_{\alpha\beta}\,\varpi_{\alpha\eta} = 0.$$

I may take the opportunity of showing how this formula can be verified in the case where α, β, and η are co-cylindroidal. Using the familiar circular representation (see 6th Memoir or the German edition of the "Theory of Screws"), we proceed thus:

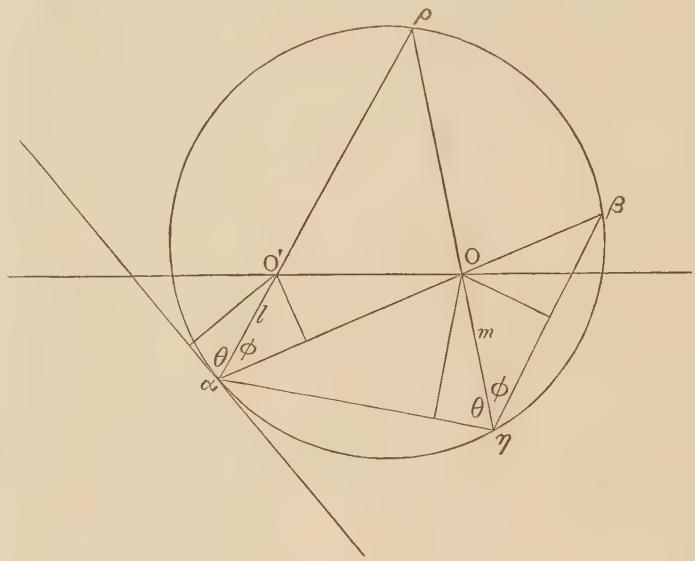

FIG. 1.

Let $O\,O'$ be the principal axis of homography where O is the pole of the axis of pitch, and O' the pole of the axis of inertia. Then, as the virtual coefficient of two screws is proportional to the perpendicular from O on their chord, and as $u_{\alpha\beta}$ is proportional to the perpendicular from O' on their chord, we easily see that

$$\varpi_{\alpha\eta} : \varpi_{\beta\eta} :: m \sin\theta : m \sin\phi,$$

and

$$u_{\alpha\beta} : u_{\alpha\alpha} :: l \sin\phi : l \sin\theta,$$

whence

$$u_{\alpha\alpha}\,\varpi_{\beta\eta} = u_{\alpha\beta}\,\varpi_{\alpha\eta}.$$

We may generalize the matter further, as follows:—

Let $\eta_1, \ldots \eta_n$ be the co-ordinates of a screw η, which is the polar of α with regard to a function U; then, the reference being as usual to a set of co-reciprocals, we have

$$H\eta_1 = \frac{1}{p_1}\frac{dU}{d\alpha},$$

&c. &c.

$$H\eta_n = \frac{1}{p_n}\frac{dU}{d\alpha_1}.$$

As before we find

$$H\varpi_{\eta\alpha} = 2\,U_{\alpha\alpha}\,;\;\; H\varpi_{\eta\beta} = 2\,U_{\alpha\beta},$$

whence, eliminating H,

$$U_{\alpha\beta}\varpi_{\alpha\eta} = U_{\alpha\alpha}\varpi_{\beta\eta}.$$

Returning to the case of the constrained rigid body, we can present the formula

$$\frac{\varpi_{\alpha\eta}\,\varpi_{\alpha\xi}}{u_{\alpha\alpha}} = \frac{\varpi_{\beta\eta}\,\varpi_{\beta\xi}}{u_{\beta\beta}},$$

in a still more compact manner.

Let us denote by the symbol $\dot{\alpha}$ the twist velocity generated around α by the influence of an impulsive wrench of unit magnitude on η.

From previous investigations in the "Theory of Screws" (see German edition, p. 431), we know that

$$\dot{\alpha} : \dot{\beta} :: \frac{\varpi_{\eta\alpha}}{u_{\alpha\alpha}} : \frac{\varpi_{\beta\xi}}{u_{\beta\beta}},$$

whence we obtain the concise result,

$$\dot{\alpha}\,\varpi_{\alpha\xi} = \dot{\beta}\,\varpi_{\beta\eta}.$$

Let us now suppose a third screw γ, of which ζ is the impulsive screw; and if $\dot{\gamma}$ have the meaning already defined, we have the following equations:—

$$\dot{\alpha}\;\varpi_{\alpha\zeta} = \dot{\gamma}\,\varpi_{\gamma\eta},$$

and

$$\dot{\beta}\,\varpi_{\beta\zeta} = \dot{\gamma}\,\varpi_{\gamma\xi}$$

Hence we obtain by eliminating $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$,

$$\varpi_{\alpha\xi}\,\varpi_{\beta\zeta}\,\varpi_{\gamma\eta} = \varpi_{\alpha\zeta}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi}.$$

This is a singular result, involving only the virtual coefficients, and it must hold good in either a free or a constrained system for any two systems of screws, whereof every corresponding pair stand in the relation of pole and polar to any function of the second order. It must now receive further development.

IV.—ON CHIASTIC HOMOGRAPHY.

In several of the preceding memoirs, and indeed throughout the whole Theory of Screws, I have had occasion to discuss the important physical meaning of homographic screw systems. The species of homography that presents itself in these physical inquiries is not the most general. We find that there is a certain specialization of the general conceptions of homography. This specialization is of a remarkable character, and it seems worthy of detailed treatment.

The Case of a System with Two Degrees of Freedom.—For the representation of the screws on a cylindroid the plane representation of that system is particularly convenient: we shall therefore adopt it for the purpose of illustrating the particular type of homography with which we are now concerned.

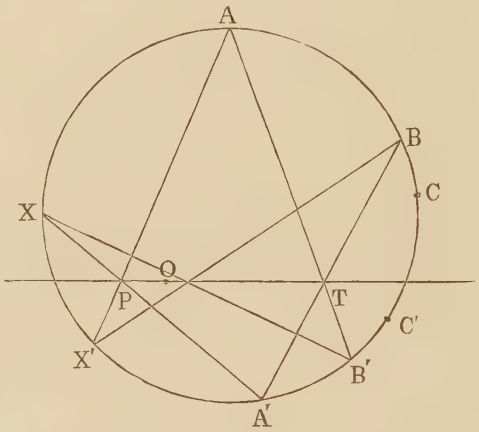

FIG. 2.

Let A, B, C, X, &c., and A', B', C', X', &c., be two homographic systems of points on a circle. These correspond respectively to two homographic systems of screws on the cylindroid. Let P, T be the axis of homography. Then it is known, from geometrical principles, that if any two pairs, such as A, A' and B, B', be taken, then the lines AB', BA' intersect on a straight line, which is the axis of the homography.

In general this axis may occupy any position whatever; if, however, it should pass through O, the pole of the axis of pitch, then the homography will assume that special type which it is our object to investigate.

In the first place, we may notice that under these circumstances the homography possesses the following characteristic:—

Let A, B be two screws, and A', B' their two correspondents; then, if A be reciprocal to B', B must be reciprocal to A'.

For in this case AB' must pass through O, and therefore BA' must pass through O also, *i. e.* B and A' must be reciprocal.

This cross relation suggests a name for the particular species of homography now before us. The form of the letter χ suggests so naturally the kind of relation, that I have found it convenient to refer to this type of homography as *Chiastic.* No doubt, in the present illustration I am only discussing the case of two degrees of freedom, but we shall presently see that chiastic homography is of import throughout the whole Theory of Screws.

The geometrical conception that the homographic axis should pass through the pole of the axis of pitch is, indeed, one of such clearness, that little more is necessary so far as the two-system is concerned. As, however, the subject will appear in a much wider aspect presently, I desire to set forth the metric relations which must be fulfilled by the members of two homographic systems of screws which possess the required property.

It is known that in the circular representation the virtual coefficients of two screws is proportional to the perpendicular distance of their chord from the pole of the axis of pitch.

Let α, β, γ be three screws of one system, and let η, ξ, ζ be the three corresponding screws, and, as usual, let $\varpi_{\alpha\xi}$ represent the virtual coefficient of α and ξ. Then we have to show that whenever the homography is chiastic the following relation is satisfied:—

$$\varpi_{\alpha\xi}\,\varpi_{\beta\zeta}\,\varpi_{\gamma\eta} = \varpi_{\alpha\zeta}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi}.$$

This is geometrically demonstrated when the following theorem is proved:—

If six points be inscribed on a circle, then the continued product of the three perpendiculars let fall from any point in a Pascal line formed from these six points upon three alternate sides of the corresponding hexagon is equal to the continued product of the three perpendiculars let fall from the same point on the other three sides.

Let $\alpha\ \alpha'$, $\beta\ \beta'$, $\gamma\ \gamma'$ be the three pairs of rays, and write the equation

$$\alpha\beta\gamma = \alpha'\beta'\gamma';$$

then this represents a cubic curve through the nine points $\alpha\ \alpha'$, $\alpha\ \beta'$, &c.

There can only be one such curve, and we know that it must break up into the ray through $\alpha\ \alpha'$, $\beta\ \beta'$, and $\gamma\ \gamma'$, and the circle through the other six points. Every point on the cubic must possess the property implied in the equation $\alpha\beta\gamma = \alpha'\beta'\gamma'$, and as the Pascal line is a part of the cubic, the required theorem is proved.

This fundamental property of two chiastic homographic systems on a cylindroid can be proved in a direct manner without the help of the circular representation, as follows :—

The several screws on the cylindroid can be defined by the tangents of the angles which they make with the principal screw. If $\tan\theta$, $\tan\theta'$ be two corresponding screws in the two homographic systems, then, for homography of quite a general type, we have the condition expressed by an equation of the type,

$$0 = A \tan\theta \tan\theta' + B \tan\theta + C \tan\theta' + D.$$

This homography contains three arbitrary constants. It is too general to denote chiastic homography. We must, therefore, first discover in what manner it is to receive the necessary specialization. The simplest characteristic of chiastic homography is, that there shall be a pair of screws, θ, θ', and another pair, ϕ, ϕ', so that both θ shall be reciprocal to ϕ', and θ' reciprocal to ϕ. The expression of this will provide the conditions which abate by unity the number of arbitrary coefficients in the equation of homography.

We may note that if this relation holds good for any set of two pairs, θ, θ' and ϕ, ϕ', then it will hold good for any other set of two pairs, that if θ be reciprocal to ϕ', then ϕ will be reciprocal to θ'. This simplifies the discovery of the necessary conditions. If θ and ϕ' correspond to reciprocal screws, then, if p_α and p_β be the principal screws on the cylindroid,

$$p_\alpha \tan\theta \tan\beta' + p_\beta = 0.$$

Take the case of $\tan\theta = 0$, then $\tan\phi' = \infty$, the homographic equation shows

$$\tan\theta' = -\frac{D}{C}, \quad \tan\phi = -\frac{C}{A}.$$

Expressing, then, that θ' and ϕ are reciprocal, we get

$$p_\alpha D + p_\beta A = 0.$$

This is at once the necessary and the sufficient condition that the homography be chiastic. We may thus express the general equation, as follows:

$$0 = p_\alpha \tan\theta \tan\theta' - p_\beta + B \tan\theta + C \tan\theta'.$$

It will now be easy to deduce the required relation between the virtual coefficients, for, we have

$$\begin{aligned}
\varpi_{\theta\phi'} &= p_\alpha \sin\theta \sin\phi' + p_\beta \cos\theta \cos\phi', \\
&= \cos\theta \cos\phi' (p_\alpha \tan\theta \tan\phi' + p_\beta), \\
&= \cos\theta \cos\phi' \left(p_\alpha \tan\theta \frac{p_\beta - B \tan\phi}{p_\alpha \tan\phi + C} + p_\beta\right), \\
&= \frac{\cos\theta \cos\phi'}{p_\alpha \tan\phi + C}\Big(p_\alpha p_\beta (\tan\theta + \tan\phi) + Cp_\beta - Bp_\alpha \tan\theta \tan\phi), \\
&= \frac{\cos\theta \cos\phi \cos\phi}{p_\alpha \sin\phi + C \cos\phi} U_{\theta\phi},
\end{aligned}$$

where we denote the factor in the bracket by $U_{\theta\phi}$.

Similarly,

$$\varpi_{\theta\psi'} = \frac{\cos\theta \cos\psi \cos\psi'}{p_\alpha \sin\psi + C \cos\psi} U_{\theta\phi},$$

$$\varpi_{\phi\theta'} = \frac{\cos\phi \cos\theta \cos\theta'}{p_\alpha \sin\theta + C \cos\theta} U_{\theta\phi},$$

$$\varpi_{\phi\psi'} = \frac{\cos\phi \cos\psi \cos\psi'}{p_\alpha \sin\psi + C \cos\psi} U_{\phi\psi},$$

$$\varpi_{\psi\theta'} = \frac{\cos\psi \cos\theta \cos\theta'}{p_\alpha \sin\theta + C \cos\theta} U_{\psi\theta},$$

$$\varpi_{\psi\phi'} = \frac{\cos\psi \cos\phi \cos\phi'}{p_\alpha \sin\phi + C \cos\phi} U_{\psi\phi},$$

whence we obtain

$$\varpi_{\theta\psi'} \times \varpi_{\phi\theta'} \times \varpi_{\psi\phi'} = \frac{\cos^2\theta \cos^2\phi \cos^2\psi \cos\theta' \cos\phi' \cos\psi'}{(p_a \sin\theta + C\cos\theta)(p_a \sin\phi + C\cos\phi)(p_a \sin\psi + C\cos\psi)}$$
$$\times U_{\phi\psi}\, U_{\phi\theta}\, U_{\psi\theta},$$

obviously, the expression for $\varpi_{\theta'\psi} \times \varpi_{\phi'\theta} \times \varpi_{\psi'\phi}$ is identical with that just written. Whence we have for chiastic homography

$$\varpi_{\theta\psi'} \times \varpi_{\phi\theta'} \times \varpi_{\psi\phi'} = \varpi_{\theta'\psi} \times \varpi_{\phi'\theta} \times \varpi_{\psi'\phi}.$$

It is an essential characteristic of chiastic homography on the cylindroid that the two double screws of the system are reciprocal. For, as the homographic axis passes through the pole of the axis of pitch, the double screws which correspond to the extremities of that axis are reciprocal.

We now proceed to study the question as to the construction of chiastic homography on the cylindroid. It is just obvious that, if two corresponding pairs of screws be arbitrarily selected, it will always be possible to devise one chiastic homography of which those two pairs are corresponding members. The circular construction (fig. 2) shows this at once, for, join AB' and $A'B$, they intersect at T, then the line TO is the homographic axis, and the correspondent to X is found by drawing $A'X$, and then AX' through P.

It is instructive to consider the case of two homographic systems on two cylindroids, and to investigate when these homographies are chiastic. The fundamental theorem for the two cylindroids is thus expressed:—

Take any two screws, α and β, on one cylindroid, and any two screws, η and ξ, on the other, it will then be possible to inscribe one, and in general only one chiastic homography on the two surfaces, such that α and η shall be correspondents, and also β and ξ.

For, write the general equation

$$\varpi_{\alpha\zeta}\, \varpi_{\beta\eta}\, \varpi_{\gamma\xi} = \varpi_{\alpha\xi}\, \varpi_{\beta\zeta}\, \varpi_{\gamma\eta}.$$

If, then, α, β, η, ξ are known, and if γ be chosen arbitrarily on the first cylindroid, it will then be always possible to find one, but only one, screw ζ on the second cylindroid which satisfies the required condition.

Of course the main interest of chiastic homography arises from the

circumstance that the physical homographies presented are always of this nature. It is to be noted that, while chiastic homography is a genus in the family of homographies generally, the several kinds of physical homography are species in the chiastic genus.

If a body had two degrees of freedom expressed by a cylindroid A, and if an arbitrary cylindroid B were taken, then an impulsive wrench administered by a screw on B would make the body commence to twist about some screw on A, and the two systems of screws would have chiastic homography. If the body were given both in constitution and in position, then, of course, there would be nothing arbitrary in the choice of the corresponding screws. Suppose, however, that a screw η had been chosen arbitrarily on B to correspond to a screw on α on A, it would then be possible to design and place a rigid body so that it should begin to twist about α in consequence of the impulse on η. There would, however, be no arbitrary element remaining in the homography, for if ξ correspond to β, the position of ξ on B will be assigned by the equation

$$\frac{\varpi_{\alpha\eta}\,\varpi_{\alpha\xi}}{u_{\alpha\alpha}} = \frac{\varpi_{\beta\eta}\,\varpi_{\beta\xi}}{u_{\beta\beta}}.$$

Thus, we see that, while for homography, in general, three pairs of correspondents can be arbitrarily assigned, there can only be two pairs so assigned for chiastic homography, while for such a particular type as that which relates to impulsive screws and the corresponding instantaneous screws, only one pair can be arbitrarily chosen.

We have already had occasion, in the Theory of Screws, to remark on the curious relationship of two cylindroids when a screw can be found on either cylindroid which is reciprocal to all the screws on the other. If, for the moment, we speak of two such cylindroids as "normal," then we have the following theorem :—

Any homography of the screws on two cylindroids must be chiastic if the two cylindroids are normal.

Let α, β, γ be any three screws on one cylindroid, and η, ξ, ζ any three screws on the other; then, since the cylindroids are normal, we have

$$\varpi_{\beta\eta}\,\varpi_{\gamma\xi} = \varpi_{\beta\xi}\,\varpi_{\gamma\eta},$$
$$\varpi_{\alpha\zeta}\,\varpi_{\beta\xi} = \varpi_{\alpha\xi}\,\varpi_{\beta\zeta},$$

whence we obtain

$$0 = \varpi_{\beta\xi}\left(\varpi_{\alpha\zeta}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi} - \varpi_{\alpha\xi}\,\varpi_{\beta\zeta}\,\varpi_{\gamma\eta}\right);$$

unless therefore ξ is reciprocal to β, we must have

$$0 = \varpi_{\alpha\zeta}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi} - \varpi_{\alpha\xi}\,\varpi_{\beta\zeta}\,\varpi_{\gamma\eta}.$$

If, however, ξ had been reciprocal to β, then one of these screws (suppose β) must have been the screw on its cylindroid reciprocal to the entire group of screws on the other cylindroid. In this case we must have

$$\varpi_{\beta\eta} = 0\,; \quad \varpi_{\beta\zeta} = 0,$$

so that even in this case it would still remain true that

$$0 = \varpi_{\alpha\zeta}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi} - \varpi_{\alpha\xi}\,\varpi_{\beta\zeta}\,\varpi_{\gamma\alpha}.$$

It is, indeed, a noteworthy circumstance that, for any and every three pairs of screws on two normal cylindroids, the relation just written must be fulfilled.

In general, when two pairs of screws are given on two cylindroids, the chiastic homography betweeu the surfaces is determined. If, however, it were possible to determine the chiastic homographies having two pairs in common, then every homography is chiastic, and the cylindroids are normal.

Let α, η and β, ξ be the two pairs of correspondents, and let γ have the correspondents ζ and ζ', then we have

$$\varpi_{\alpha\zeta}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi} = \varpi_{\alpha\xi}\,\varpi_{\beta\zeta}\,\varpi_{\gamma\eta},$$

$$\varpi_{\alpha\zeta'}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi} = \varpi_{\alpha\xi}\,\varpi_{\beta\zeta'}\,\varpi_{\gamma\eta},$$

whence

$$\varpi_{\alpha\zeta}\,\varpi_{\beta\zeta'} = \varpi_{\alpha\zeta'}\,\varpi_{\beta\zeta},$$

i. e. the two cylindroids are normal.

General Conditions of Chiastic Homography.—We shall now discuss the relations of homography between two systems of screws included in the same n-system. The first point to be demonstrated is, that in such a case every pair of the double screws are reciprocal.

Take α and β as two of the double screws, and η and ξ will coincide with them; whence the general condition,

$$\varpi_{\alpha\zeta}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi} = \varpi_{\alpha\xi}\,\varpi_{\beta\zeta}\,\varpi_{\gamma\eta}$$

becomes

$$\varpi_{\alpha\zeta}\,\varpi_{\alpha\beta}\,\varpi_{\beta\gamma} = \varpi_{\alpha\beta}\,\varpi_{\beta\zeta}\,\varpi_{\alpha\gamma},$$

$$\varpi_{\alpha\beta}\left(\varpi_{\alpha\zeta}\,\varpi_{\beta\gamma} - \varpi_{\alpha\gamma}\,\varpi_{\beta\zeta}\right) = 0.$$

One or other of these factors must be zero. We have to demonstrate that in general it is impossible for

$$\varpi_{\alpha\zeta}\,\varpi_{\beta\gamma} - \varpi_{\alpha\gamma}\,\varpi_{\beta\zeta}$$

to vanish.

For, take γ reciprocal to α but not to β, then $\varpi_{\alpha\gamma} = 0$; but $\varpi_{\beta\gamma}$ is not zero, and therefore $\varpi_{\alpha\zeta}$ would have to be zero; in other words ζ must be reciprocal to α. But this cannot generally be the case, and hence the other factor must vanish, that is

$$\varpi_{\alpha\beta} = 0.$$

In like manner it can be shown that every pair of the double screws must be reciprocal.

Conversely it can be shown that if the double screws of two homographic systems are co-reciprocal, then the homography is chiastic.

Let the n-double screws of the two systems be taken as the screws of reference; then if one screw in one system be denoted by the co-ordinates

$$\alpha_1, \ldots \alpha_n,$$

its correspondent in the other system will be

$$\lambda h_1\,\alpha_1, \ldots \lambda h_n\,\alpha_n.$$

Similarly, the correspondent to

$$\beta_1, \ldots \beta_n$$

will have for its co-ordinates

$$\mu h_1\,\beta_1, \ldots \mu h_n\,\beta_n,$$

and the correspondent to

$$\gamma_1, \ldots \gamma_n$$

will have for its co-ordinates

$$\nu h_1\,\gamma_1, \ldots \nu h_n\,\gamma_n,$$

when λ, μ, ν, are the constants requisite to make the co-ordinates fulfil the fundamental conditions as to dimensions.

We thus compute

$$\varpi_{\alpha\xi} = \mu\,(p_1\,h_1\,\alpha_1\,\beta_1 + \ldots + p_n\,h_n\,\alpha_n\,\beta_n);$$

and similarly for the other terms.

Whence, by substitution, we find the following equation identically satisfied:—

$$\varpi_{\alpha\xi}\,\varpi_{\beta\zeta}\,\varpi_{\gamma\eta} = \varpi_{\alpha\zeta}\,\varpi_{\beta\eta}\,\varpi_{\gamma\xi}.$$

It may be noted that, in a 3-system, two homographies are chiastic when, in the plane representation by points, the double points of the two systems form a triangle which is self-conjugate with respect to the pitch conic.

In like manner, when two homographies in a 4-system are chiastic, the double elements in the spacial representation by points form the vertices of a tetrahedron self-conjugate with regard to the quadric corresponding to the screws of zero pitch.

V.—On a Differential Equation satisfied by the Function expressing Kinetic Energy.

Suppose that the pitch of a screw θ, about which a rigid body is twisting with a velocity $\dot{\theta}$, be augmented by δp, what effect will that produce on the kinetic energy, assuming that the twist velocity $\dot{\theta}$ remains unaltered.

The linear velocity of the body is augmented by the magnitude $\dot{\theta}\delta p$. The energy of translation is $\frac{1}{2}\,Mv^2$, and for a variation δv in the velocity the alteration in kinetic energy is $Mv\delta v$. As v is $p\dot{\theta}$ and δv is $\dot{\theta}\delta p$. Hence the effect on the kinetic energy is $M\dot{\theta}^2 p\delta p$.

But there is another method of computing the same effect on the total kinetic energy.

The addition of δp to the pitch alters the co-ordinates $\theta_1, \ldots \theta_6$ into

$$\theta_1 + \frac{\delta p}{4p_1}\frac{dR}{d\theta_1}, \ldots \theta_6 + \frac{\delta p}{4p_6}\frac{dR}{d\theta_6}.$$

and therefore, with this alteration, the kinetic energy T becomes

$$T + \frac{\delta p}{4}\left(\frac{1}{p_1}\frac{dR}{d\theta_1}\cdot\frac{dT}{d\theta_1} + \ldots + \frac{1}{p_6}\frac{dR}{d\theta_6}\cdot\frac{dT}{d\theta_6}\right),$$

whence

$$Mp_\theta \delta p \dot{\theta}^2 = \frac{\delta p}{4p_1} \cdot \frac{dR}{d\theta_1} \cdot \frac{dT}{d\theta_1} + \ldots + \frac{\delta p}{4p_6} \cdot \frac{dR}{d\theta_6} \cdot \frac{dT}{d\theta_6}.$$

If T_o be the part of the energy due to the translation only, we have

$$T_o = \tfrac{1}{2} M p_\theta^2 \dot{\theta}^2;$$

whence

$$\frac{1}{p_\theta} = \frac{1}{2T_0}\left(\frac{1}{4p_1} \frac{dR}{d\theta_1} \cdot \frac{dT}{d\theta_1} + \ldots + \frac{1}{4p_6} \cdot \frac{dR}{d\theta_6} \cdot \frac{dT}{d\theta_6}\right).$$

Or we may express the result in this way. Let T be the kinetic energy of the body; if the twist velocity be units, and the mass be unity, then

$$p_\theta = DT;$$

where D is the operation thus defined,

$$D = \frac{1}{4p_1} \frac{dR}{d\theta_1} \cdot \frac{d}{d\theta_1} + \ldots + \frac{1}{4p_6} \cdot \frac{dR}{d\theta_6} \cdot \frac{d}{d\theta_6}.$$

This is a singular result. It is really an operation which, applied to the kinetic energy, yields the pitch of the screw about which the system is twisting.

Another result may be noticed.

If k be the radius of gyration around θ,

$$T = \tfrac{1}{2} M\dot{\theta}^2 (p_\theta^2 + k^2),$$

$$\frac{dT}{d\theta_1} = M\dot{\theta}^2 p_\theta \cdot \frac{dp_\theta}{d\theta_1} + \tfrac{1}{2} M\dot{\theta}^2 \frac{dk^2}{d\theta_1},$$

$$\frac{dp_\theta}{d\theta_1} = 2p_1\theta,$$

whence

$$\frac{dT}{d\theta_1} = 2M\dot{\theta}^2 p_\theta p_1 \theta_1 + \tfrac{1}{2} M\dot{\theta}^2 \frac{dk^2}{d\theta_1},$$

and, similarly,

$$\frac{dT}{d\theta_2} = 2M\dot{\theta}^2 p_\theta p_2 \theta_2 + \tfrac{1}{2} M\dot{\theta}^2 \frac{dk^2}{d\theta_2};$$

substituting these values in the equation

$$Mp_\theta \dot{\theta}^2 = \frac{1}{4p_1} \cdot \frac{dR}{d\theta_1} \frac{dT}{d\theta_1} + \ldots + \frac{1}{4p_6} \cdot \frac{dR}{d\theta_6} \cdot \frac{dT}{d\theta_6},$$

we get

$$Mp\,\dot{\theta}^2 = 2\,M\dot{\theta}^2 p_0 \left(\frac{1}{4p_1}\frac{dR}{d\theta_1}p_1\theta_1 + \ldots + \frac{1}{4p_6}\cdot\frac{dR}{d\theta_6}p_6\,\theta_6\right),$$

$$+\tfrac{1}{2}\,M\dot{\theta}^2\left(\frac{1}{4p_1}\frac{dR}{d\theta_1}\cdot\frac{dk^2}{d\theta_1} + \ldots + \frac{1}{4p_1}\cdot\frac{dR}{d\theta_6}\frac{dk^2}{d\theta_6}\right),$$

but
$$\theta_1\frac{dR}{d\theta_1} + \ldots + \theta_6\frac{dR}{d\theta_6} = 2R = 2,$$

whence we have the remarkable result,

$$\frac{1}{p_1}\frac{dR}{d\theta_1}\cdot\frac{dk^2}{d\theta_1} + \ldots + \frac{1}{p_6}\cdot\frac{dR}{d\theta_6}\cdot\frac{dk^2}{d\theta_6} = 0,$$

This is a partial differential equation which must be satisfied by the square of the radius of gyration.

VI.—A GEOMETRICAL NOTE.

In the study of the 3-system the following theorem is useful:—

Any ray η which crosses at right angles two screws of a three-system α and β is the seat of a screw reciprocal to the system.

For, draw the cylindroid α, β, then of course η, whatever be its pitch, is reciprocal to all the screws on this cylindroid. Through any point P on η there are two screws of the system which lie on the cylindroid, and there must be a third screw γ of the system through P, which, certainly, does not lie on the cylindroid. If, therefore, we give η a pitch $-p_\gamma$, it must be reciprocal to the 3-system.

In general, one screw of a three-system can be found which intersects at right angles any screw η in space.

For η must, of course, cut each of the quadrics containing the equally pitched screws in two points. Take, for example, the quadric with screws of pitch p. There are, therefore, two screws, α and β of pitch p belonging to the system which intersect η. The cylindroid α, β must belong to the system, and from the known property of the cylindroid the ray η, which crosses the two equal pitch screws, must cross at right angles some third screw γ on this cylindroid, but this belongs to the 3-system, and therefore the theorem has been proved.